

ENERGY-MOMENTUM TENSOR OF A CASIMIR APPARATUS IN A WEAK GRAVITATIONAL FIELD: SCALAR CASE

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Abstract

Recent work in the literature had evaluated the energy-momentum tensor of a Casimir apparatus in a weak gravitational field, for an electromagnetic field subject to perfect conductor boundary conditions on parallel plates. The Casimir apparatus was then predicted to experience a tiny push in the upwards direction, and the regularized energy-momentum tensor was found to have a trace anomaly. The latter, unexpected property made it compelling to assess what happens in a simpler case. For this purpose, the present paper studies a free, real massless scalar field subject to homogeneous Dirichlet conditions on the parallel plates. Working to first order in the constant gravity acceleration, the resulting regularized and renormalized energy-momentum tensor is found to be covariantly conserved, while the trace anomaly vanishes if the massless scalar field is conformally coupled to gravity. Conformal coupling also ensures a finite Casimir energy and finite values of the pressure upon parallel plates.

I. INTRODUCTION

Ever since Casimir discovered that suitable differences of zero-point energies of the quantized electromagnetic field can be made finite and produce measurable effects [1], several efforts have been produced to understand the physical implications and applications of this property [2, 3, 4, 5, 6]. In particular, we are here concerned with the recent theoretical discovery that Casimir energy gravitates [7, 8, 9, 10]. In Ref. [9], some of us proved this as part of an investigation which led to the evaluation of the energy-momentum tensor of a Casimir apparatus in a weak gravitational field. In that investigation, the functional integral quantization of Maxwell theory was applied, with perfect conductor boundary conditions on parallel plates at distance a from each other. On using Fermi–Walker coordinates, where the (x_1, x_2) coordinates span the plates, while the $z \equiv x_3$ axis coincides with the vertical upwards direction (so that the plates have equations $z = 0$ and $z = a$, respectively), and working to first order in the constant gravity acceleration g , the spacetime metric reads as [9]

$$ds^2 = -c^2 \left(1 + \varepsilon \frac{z}{a}\right) dt^2 + dx_1^2 + dx_2^2 + dz^2 + O(|x|^2), \quad (1.1)$$

where $\varepsilon \equiv \frac{2ga}{c^2}$. The resulting regularized (with point-split method) and renormalized energy-momentum tensor $\langle T^{\mu\nu} \rangle$ was found to be covariantly conserved, with photon and ghost Green functions obeying the mixed boundary conditions of the problem and satisfying the Ward identities [9]. All of this would have been completely reassuring, had it not been for the fact that such a $\langle T^{\mu\nu} \rangle$ was found to have a trace anomaly

$$\langle T_\mu^\mu \rangle = a^{-3} f\left(\frac{z}{a}\right), \quad (1.2)$$

where

$$f\left(\frac{z}{a}\right) \equiv \frac{\pi}{360} \frac{hg}{c} \left(\frac{z}{a} - \frac{15}{2\pi} \frac{\cos(\pi z/a)}{\sin^3(\pi z/a)}\right). \quad (1.3)$$

As is by now well known, even though the classical action is invariant under conformal rescalings of the metric, a trace anomaly may arise because some counterterms may occur which fail to possess the same invariances as the classical action (see Comments on chapter 28 in Ref. [11]). In four spacetime dimensions, such counterterms are quadratic in Riemann, Ricci and scalar curvature of the background. Moreover, if the boundary is nonempty, the trace anomaly receives further contributions from local invariants which are cubic in the

extrinsic-curvature tensor K_{ij} of the boundary, i.e.

$$K_i^j K_j^l K_l^i, K_i^i K_{lm} K^{lm}, (K_i^i)^3.$$

All of this is made precise by the proportionality between trace anomaly and the local heat-kernel coefficient a_2 [12] obtained by studying the heat equation for an operator of Laplace type on a Riemannian manifold with boundary [13, 14, 15].

Our analysis of the energy-momentum tensor, however, was Lorentzian rather than Euclidean. We obtained the Hadamard function as twice the imaginary part of the Feynman Green function, and then used the point-split method to obtain $\langle T^{\mu\nu} \rangle$. It became therefore important to check the consistency of the above procedure by studying a simpler problem, and for this purpose we have here focused on a scalar problem. Our “Casimir” apparatus involves a massless scalar field in curved background, subject to homogeneous Dirichlet conditions on parallel plates with mutual separation a .

Section II is a summary of basic properties of a free massless scalar field in curved space-time: action functional, classical energy-momentum tensor and its regularized expression. Section III evaluates the Feynman Green function up to first order in ε . Section IV obtains the regularized and renormalized energy-momentum tensor of the quantum theory, while Sec. V evaluates the Casimir energy. Discussion of the results and open problems are presented in Sec. VI.

II. FREE MASSLESS SCALAR FIELD IN CURVED SPACETIME

The action functional for a free massless scalar field ϕ coupled to gravity reads as

$$S = -\frac{1}{2} \int \left(\phi_{;\mu} \phi^{;\mu} + \xi R \phi^2 \right) \sqrt{-g} d^4x, \quad (2.1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R is the scalar curvature and ξ is a real parameter taking the value $\frac{1}{6}$ in the case of conformal coupling and 0 in the case of minimal coupling. On requiring stationarity of the action functional under variations of ϕ , one obtains the field equation

$$(\square - \xi R)\phi = 0, \quad (2.2)$$

where $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the wave operator.

The classical energy-momentum tensor is obtained from functional differentiation of the classical action with respect to the metric, i.e.

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \quad (2.3)$$

If use is made of the standard variational identities

$$\begin{aligned} \delta g^{\mu\nu} &= -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}, \quad \delta g = g g^{\mu\nu} \delta g_{\mu\nu}, \\ \delta R &= -R^{\mu\nu} \delta g_{\mu\nu} + g^{\mu\rho} g^{\nu\sigma} (\delta g_{\rho\sigma;\mu\nu} - \delta g_{\nu\sigma;\rho\mu}), \end{aligned}$$

and by introducing the anticommutator

$$[\phi(x), \phi(y)]_+ \equiv \phi(x)\phi(y) + \phi(y)\phi(x), \quad (2.4)$$

one finds eventually [16]

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}(1 - 2\xi)[\phi^{;\mu}, \phi^{;\nu}]_+ + \left(\xi - \frac{1}{4}\right) g^{\mu\nu} [\phi_{;\sigma}, \phi^{;\sigma}]_+ - \xi [\phi^{;\mu\nu}, \phi]_+ \\ &\quad + \xi g^{\mu\nu} [\phi_{;\sigma}{}^{\sigma}, \phi]_+ + \frac{\xi}{2} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) [\phi, \phi]_+. \end{aligned} \quad (2.5)$$

At this stage, the point-split method comes into play, and we re-express every anticommutator in (2.5) according to

$$[\phi^{;\mu}, \phi^{;\nu}]_+ = \lim_{x' \rightarrow x} \frac{1}{2} \left\{ [\phi^{;\mu'}, \phi^{;\nu}]_+ + [\phi^{;\mu}, \phi^{;\nu'}]_+ \right\}, \quad (2.6)$$

$$[\phi^{;\mu\nu}, \phi]_+ = \lim_{x' \rightarrow x} \frac{1}{2} \left\{ [\phi^{;\mu'\nu'}, \phi]_+ + [\phi^{;\mu\nu}, \phi']_+ \right\}, \quad (2.7)$$

$$[\phi, \phi]_+ = \lim_{x' \rightarrow x} [\phi, \phi']_+. \quad (2.8)$$

On introducing the Hadamard two-point function (hereafter, the brackets $\langle \rangle$ denote the (vacuum) expectation value)

$$H(x, x') = \langle [\phi(x), \phi(x')]_+ \rangle, \quad (2.9)$$

we get (cf. [16])

$$\begin{aligned} \langle T^{\mu\nu} \rangle &= \lim_{x' \rightarrow x} \left[\frac{(1 - 2\xi)}{4} \left(H^{;\mu'\nu} + H^{;\mu\nu'} \right) + \left(\xi - \frac{1}{4} \right) g^{\mu\nu} H_{;\sigma}{}^{\sigma'} \right. \\ &\quad \left. - \frac{\xi}{2} \left(H^{;\mu\nu} + H^{;\mu'\nu'} \right) + \frac{\xi}{2} g^{\mu\nu} \left(H_{;\sigma}{}^{\sigma} + H_{;\sigma'}{}^{\sigma'} \right) + \frac{\xi}{2} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) H \right]. \end{aligned} \quad (2.10)$$

Note that the use of general equations has payed off: if we now focus on Ricci-flat spacetimes, the effect of ξ remains, which would not be obvious if Ricci-flatness were imposed in (2.1). Hereafter, on taking the coincidence limit, we need of course the geodesic parallel displacement bivector $P_{\nu'}^{\mu}$, which performs parallel displacement of vectors along the geodesic from x' to x . In general, it is defined by the differential equations [9]

$$\sigma^{;\rho} P_{\nu';\rho}^{\mu} = \sigma^{;\tau'} P_{\nu';\tau'}^{\mu} = 0, \quad (2.11)$$

$\sigma(x, x')$ being the Ruse–Synge world function [17], equal to half the geodesic distance between x and x' , jointly with the coincidence limit

$$\lim_{x' \rightarrow x} P_{\nu'}^{\mu} \equiv [P_{\nu'}^{\mu}] = \delta_{\nu}^{\mu}. \quad (2.12)$$

Equation (2.11) means that the covariant derivatives of $P_{\nu'}^{\mu}$ vanish in the directions tangent to the geodesic joining x and x' . Thus, the bivector $P_{\nu'}^{\mu}$, when acting on a vector $B^{\nu'}$ at x' , gives the vector \overline{B}^{μ} , which is obtained by parallel transport of $B^{\nu'}$ to x along the geodesic connecting x and x' , i.e.

$$\overline{B}^{\mu} = P_{\nu'}^{\mu} B^{\nu'}. \quad (2.13)$$

In our problem, we have from Eq. (1.1) the metric tensor

$$g_{\mu\nu} = \text{diag} \left(-1 - \varepsilon \frac{z}{a}, 1, 1, 1 \right), \quad (2.14)$$

with contravariant form

$$g^{\mu\nu} \sim \text{diag} \left(-1 + \varepsilon \frac{z}{a}, 1, 1, 1 \right) + \mathcal{O}(\varepsilon^2). \quad (2.15)$$

The orthonormal tetrad e_{μ}^a relating the spacetime metric $g_{\mu\nu}$ to the Minkowski metric η_{ab} according to

$$g_{\mu\nu} = e_{\mu}^a e_{\nu}^b \eta_{ab} \quad (2.16)$$

is thus given by

$$e_{\mu}^0 = \sqrt{-g_{00}} \delta_{\mu}^0, \quad e_{\mu}^i = \delta_{\mu}^i \quad \forall i = 1, 2, 3. \quad (2.17)$$

The bivector $P_{\nu'}^{\mu}$, hence reads as [18]

$$P_{\nu'}^{\mu} = g^{\mu\rho} \eta_{ab} e_{\rho}^b e_{\nu'}^a \sim \text{diag} \left(1 + \frac{\varepsilon}{2a} (z' - z), 1, 1, 1 \right) + \mathcal{O}(\varepsilon^2), \quad (2.18)$$

while

$$P_{\mu}^{\nu'} = g_{\mu\rho} g^{\nu'\beta'} P_{\beta'}^{\rho} \sim \text{diag} \left(1 + \frac{\varepsilon}{2a} (z - z'), 1, 1, 1 \right) + \mathcal{O}(\varepsilon^2). \quad (2.19)$$

The corresponding geodesic is described by the equations

$$t = \text{constant}, \quad x_1, x_2 = \text{constant}, \quad (2.20)$$

$$z_\psi \equiv \psi(z' - z) + z, \quad \psi \in [0, 1]. \quad (2.21)$$

III. FEYNMAN GREEN FUNCTION TO ZEROth AND FIRST ORDER

The equations II were rather general, whereas now we consider, following the introduction and our previous work [9, 10], Fermi–Walker coordinates for a system of two parallel plates in a weak gravitational field. To first order in the ε parameter of Secs. I and II, the only nonvanishing Christoffel symbols associated with this metric are therefore

$$\Gamma_{30}^0 = \Gamma_{03}^0 = \frac{\varepsilon}{2(a + \varepsilon z)} \sim \frac{\varepsilon}{2a} + O(\varepsilon^2), \quad \Gamma_{00}^3 \sim \frac{\varepsilon}{2a} + O(\varepsilon^2). \quad (3.1)$$

We are now in a position to evaluate the scalar counterpart of the analysis in Ref. [9], i.e. we compute the wave operator \square , the Feynman Green function of the hyperbolic operator $(\square - \xi R)$, and eventually the Hadamard function and the regularized energy-momentum tensor.

Indeed, a Green function of the operator ruling the field equation (2.2) obeys the differential equation

$$(\square - \xi R)G(x, x') = -\frac{\delta(x, x')}{\sqrt{-g}}. \quad (3.2)$$

The Feynman Green function G_F is the unique symmetric complex-valued Green function which obeys the relation [19]

$$\delta G = G \delta F G,$$

where F is the invertible operator obtained from variation of the action functional with respect to the field. This definition is well suited for the purpose of defining the Feynman Green function even when asymptotic flatness does not necessarily hold [19].

In our first-order expansion in the ε parameter, the scalar curvature gives vanishing contribution to Eq. (3.4), which therefore takes the form (hereafter $\square^0 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$)

$$\left(\square^0 + \frac{\varepsilon z}{(a + \varepsilon z)} \frac{\partial^2}{\partial t^2} + \Gamma_{00}^3 \frac{a}{(a + \varepsilon z)} \frac{\partial}{\partial z} \right) G(x, x') = -\frac{\delta(x, x')}{\sqrt{-g}}. \quad (3.3)$$

We now follow our work in Ref. [9] and assume that the Feynman Green function admits the asymptotic expansion

$$G_F(x, x') \sim G^{(0)}(x, x') + \varepsilon G^{(1)}(x, x') + O(\varepsilon^2). \quad (3.4)$$

It is a nontrivial property that this asymptotics should hold at small ϵ *for all* x, x' , no matter how close or distant from each other are the two points. Its existence is proved by the calculations described hereafter. Indeed, by insertion of (3.4) into (3.3) we therefore obtain, picking out terms of zeroth and first order in ϵ , the pair of differential equations

$$\square^0 G^{(0)}(x, x') = J^{(0)}(x, x'), \quad (3.5)$$

$$\square^0 G^{(1)}(x, x') = J^{(1)}(x, x'), \quad (3.6)$$

having set

$$J^{(0)}(x, x') \equiv -\delta(x, x'), \quad (3.7)$$

$$J^{(1)}(x, x') \equiv \frac{z}{2a} \delta(x, x') - \left(\frac{z}{a} \frac{\partial^2}{\partial t^2} + \frac{1}{2a} \frac{\partial}{\partial z} \right) G^{(0)}(x, x'). \quad (3.8)$$

Our boundary conditions are Dirichlet in the spatial variable z . Since the full Feynman function $G_F(x, x')$ is required to vanish at $z = 0, a$, this implies the following homogeneous Dirichlet conditions on the zeroth and first-order terms:

$$G^{(0)}(x, x') \Big|_{z=0,a} = 0, \quad (3.9)$$

$$G^{(1)}(x, x') \Big|_{z=0,a} = 0. \quad (3.10)$$

To solve Eqs. (3.5) and (3.6), we perform a Fourier analysis of $G^{(0)}$ and $G^{(1)}$, which remains meaningful in a weak gravitational field [9], by virtue of translation invariance. In such an analysis we separate the z variable, i.e. we write (cf. [9])

$$G^{(0)}(x, x') = \int \frac{dk^0 d\vec{k}_\perp}{(2\pi)^3} \gamma^{(0)}(z, z') e^{i\vec{k}_\perp \cdot (\vec{x}_\perp - \vec{x}'_\perp) - ik^0(x_0 - x'_0)}, \quad (3.11)$$

and similarly for $G^{(1)}(x, x')$, with a “reduced Green function” $\gamma^{(1)}(z, z')$ in the integrand as a counterpart of the zeroth-order Green function $\gamma^{(0)}(z, z')$ in (3.11). Equations (3.5) and (3.6) lead therefore to the following equations for reduced Green functions (hereafter $\lambda \equiv \sqrt{k_0^2 - k_\perp^2}$):

$$\left(\frac{\partial^2}{\partial z^2} + \lambda^2 \right) \gamma^{(0)}(z, z') = -\delta(z, z'), \quad (3.12)$$

$$\left(\frac{\partial^2}{\partial z^2} + \lambda^2 \right) \gamma^{(1)}(z, z') = \frac{z}{2a} \delta(z, z') + \left(\frac{z}{a} k_0^2 - \frac{1}{2a} \frac{\partial}{\partial z} \right) \gamma^{(0)}(z, z'). \quad (3.13)$$

By virtue of the Dirichlet conditions (3.11), $\gamma^{(0)}$ reads as

$$\gamma^{(0)}(z, z') = -\frac{\sin(\lambda z_{<}) \sin(\lambda(z_{>} - a))}{\lambda \sin(\lambda a)}, \quad (3.14)$$

where $z_{<} \equiv \min(z, z')$, $z_{>} \equiv \max(z, z')$. The evaluation of the reduced Green function $\gamma^{(1)}$ is slightly more involved. For this purpose, we distinguish the cases $z < z'$ and $z > z'$, and find the two equations

$$\left(\frac{\partial^2}{\partial z^2} + \lambda^2\right) \gamma_{\pm}^{(1)}(z, z') = j_{\pm}^{(1)}(z, z'), \quad (3.15)$$

where

$$j_{-}^{(1)} = \frac{1}{2a} \frac{\lambda \cos(\lambda z) - 2zk_0^2 \sin(\lambda z)}{\lambda \sin(\lambda a)} \sin(\lambda(z' - a)) \text{ if } z < z', \quad (3.16)$$

$$j_{+}^{(1)} = \frac{1}{2a} \frac{\lambda \cos(\lambda(z - a)) - 2zk_0^2 \sin(\lambda(z - a))}{\lambda \sin(\lambda a)} \sin(\lambda z') \text{ if } z > z'. \quad (3.17)$$

We have therefore two different solutions in the intervals $z < z'$ and $z > z'$. In this case the differential equation (3.13) is solved by imposing the matching condition

$$\gamma_{-}^{(1)}(z', z') = \gamma_{+}^{(1)}(z', z') \quad (3.18)$$

jointly with the jump condition

$$\left. \frac{\partial}{\partial z} \gamma_{+}^{(1)} \right|_{z=z'} - \left. \frac{\partial}{\partial z} \gamma_{-}^{(1)} \right|_{z=z'} = \frac{z'}{2a}. \quad (3.19)$$

Equation (3.18) is just the continuity requirement of the reduced Green function $\gamma^{(1)}(z, z')$ at $z = z'$, while Eq. (3.19) can be obtained by integrating Eq. (3.13) in a neighborhood of z' , since

$$\lim_{\epsilon \rightarrow 0} \left. \frac{\partial}{\partial z} \gamma^{(1)} \right|_{z'=z'-\epsilon}^{z'=z'+\epsilon} = \lim_{\epsilon \rightarrow 0} \int_{z'-\epsilon}^{z'+\epsilon} \frac{z}{2a} \delta(z, z') dz = \frac{z'}{2a}. \quad (3.20)$$

Bearing in mind Eq. (3.14) we can therefore write, for all z, z' ,

$$\begin{aligned} \gamma^{(1)}(z, z') &= \frac{1}{4a\lambda^2} \left\{ \left[(k_0^2 - \lambda^2)(z + z') - k_0^2 \left(z^2 \frac{\partial}{\partial z} + z'^2 \frac{\partial}{\partial z'} \right) \right] \gamma^{(0)}(z, z') \right. \\ &\quad \left. - k_0^2 a^2 \frac{\sin(\lambda z) \sin(\lambda z')}{\sin^2(\lambda a)} \right\}. \end{aligned} \quad (3.21)$$

IV. REGULARIZED AND RENORMALIZED ENERGY-MOMENTUM TENSOR

In the previous section we have focused on the Feynman Green function G_F because it is then possible to develop a recursive scheme for the evaluation of its asymptotic expansion at small ε . However, as is clear from Eq. (2.10), we eventually need the Hadamard function $H(x, x')$, which is obtained as [9]

$$H(x, x') \equiv 2\text{Im}G_F(x, x') \sim 2\text{Im}(G^{(0)}(x, x') + \varepsilon G^{(1)}(x, x')) + \mathcal{O}(\varepsilon^2). \quad (4.1)$$

The coincidence limits in (2.10), with the help of (2.13) and (2.18), make it necessary to perform the replacements

$$H_{;\mu'\nu} + H_{;\mu\nu'} \rightarrow P_{\mu}^{\mu'} H_{;\mu'\nu} + P_{\nu}^{\nu'} H_{;\mu\nu'}, \quad H_{;\sigma}^{\sigma'} \rightarrow g^{\sigma\rho} P_{\rho}^{\rho'} H_{;\sigma\rho'}, \quad H_{;\mu'\nu'} \rightarrow P_{\mu}^{\mu'} P_{\nu}^{\nu'} H_{;\mu'\nu'}. \quad (4.2)$$

Hence we get the asymptotic expansion at small ε of the regularized energy-momentum tensor according to (hereafter we evaluate its covariant, rather than contravariant, form)

$$\langle T_{\mu\nu} \rangle \sim \langle T_{\mu\nu}^{(0)} \rangle + \varepsilon \langle T_{\mu\nu}^{(1)} \rangle + O(\varepsilon^2), \quad (4.3)$$

where, on defining $s \equiv \pi z/a$, $s' \equiv \pi z'/a$, we find

$$\begin{aligned} \langle T_{\mu\nu}^{(0)} \rangle = & \left[-\frac{\pi^2}{1440a^4} - \lim_{s' \rightarrow s} \frac{\pi^2}{2a^4(s-s')^4} \right] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\ & + \left(\xi - \frac{1}{6} \right) \frac{\pi^2}{8a^4} \left[\frac{3 - 2\sin^2 s}{\sin^4 s} \right] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \langle T_{00}^{(1)} \rangle = & \frac{\pi}{1440a^4 \sin^4 s} \left[\frac{311}{40} \pi - \frac{637}{40} s + \frac{1}{10} (43\pi - 81s) \cos 2s \right. \\ & + \left. \frac{s - 3\pi}{40} \cos 4s + 5 \sin 2s + 2(\pi - s)s(\sin 2s - 6 \cot s) \right] \\ & + \left(\xi - \frac{1}{6} \right) \frac{\pi}{48a^4 \sin^4 s} \left[2(\pi + s)(2 + \cos 2s) + \frac{5}{2} \sin 2s \right. \\ & + \left. (\pi - s)s(\sin 2s - 6 \cot s) \right] - \lim_{s' \rightarrow s} \frac{\pi s}{2a^4(s-s')^4}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \langle T_{11}^{(1)} \rangle = & \frac{\pi}{7200a^4} \left[\pi - 2s + \frac{5}{\sin^2 s} \left(2(\pi - 2s) \left(-2 + \frac{3}{\sin^2 s} \right) \right. \right. \\ & + \left. \left. \cot s \left(5 + 2(\pi - s)s - 6(\pi - s) \frac{s}{\sin^2 s} \right) \right) \right] \\ & + \left(\xi - \frac{1}{6} \right) \frac{\pi}{96a^4 \sin^5 s} \left[(11(\pi - s)s - 1) \cos s \right. \\ & + \left. ((\pi - s)s + 1) \cos 3s - 2(\pi - 2s)(3 \sin s + \sin 3s) \right], \end{aligned} \quad (4.6)$$

$$\langle T_{22}^{(1)} \rangle = \langle T_{11}^{(1)} \rangle, \quad (4.7)$$

$$\langle T_{33}^{(1)} \rangle = -\frac{\pi^2}{1440a^4} + \frac{\pi s}{720a^4} + \left(\xi - \frac{1}{6} \right) \frac{\pi}{16a^4} \frac{\cos s}{\sin^3 s}. \quad (4.8)$$

The next step of our analysis is the renormalization of the regularized energy-momentum tensor. For this purpose, following our work in Ref. [9], we subtract the energy-momentum tensor evaluated in the absence of bounding plates, i.e.

$$\langle \tilde{T}_{\mu\nu}^{(0)} \rangle = -\lim_{s' \rightarrow s} \frac{\pi^2}{2a^4(s-s')^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad (4.9)$$

and

$$\langle \tilde{T}_{\mu\nu}^{(1)} \rangle = -\lim_{s' \rightarrow s} \frac{\pi s}{2a^4(s-s')^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.10)$$

To test consistency of our results we should now check whether our regularized and renormalized energy-momentum tensor is covariantly conserved, since otherwise we would be outside the realm of quantum field theory in curved spacetime, which would be unacceptable. Indeed, the condition

$$\nabla^\mu \langle T_{\mu\nu} \rangle = 0 \quad (4.11)$$

yields, working up to first order in ε , the pair of equations

$$\frac{\partial}{\partial z} \langle T_{33}^{(0)} \rangle = 0, \quad (\varepsilon^0 \text{ term}) \quad (4.12)$$

$$\frac{\partial}{\partial z} \langle T_{33}^{(1)} \rangle + \frac{1}{2a} \left(\langle T_{00}^{(0)} \rangle + \langle T_{33}^{(0)} \rangle \right) = 0 \quad (\varepsilon^1 \text{ term}), \quad (4.13)$$

which are found to hold identically for all values of ξ in our problem.

The trace of $\langle T_{\mu\nu} \rangle$ is obtained as ($g^{\mu\nu}$ being the contravariant metric in (2.15))

$$\tau \equiv g^{\mu\nu} \langle T_{\mu\nu} \rangle \sim \eta^{\mu\nu} \langle T_{\mu\nu}^{(0)} \rangle + \varepsilon \left[\eta^{\mu\nu} \langle T_{\mu\nu}^{(1)} \rangle + \frac{z}{a} \langle T_{00}^{(0)} \rangle \right] + \mathcal{O}(\varepsilon^2), \quad (4.14)$$

from which we find a ξ -dependent part

$$\begin{aligned} \tau_\xi = & \left(\xi - \frac{1}{6} \right) \left\{ -\frac{3\pi^2(2 + \cos 2s)}{8a^4 \sin^4 s} - \varepsilon \frac{\pi}{32a^4 \sin^5 s} \left[(1 - 11(\pi - s)s) \cos s \right. \right. \\ & \left. \left. - (1 + (\pi - s)s) \cos 3s + 2(\pi - 2s)(3 \sin s + \sin 3s) \right] \right\}. \end{aligned} \quad (4.15)$$

Interestingly, the value $\xi = \frac{1}{6}$ which yields conformal invariance of the classical action (2.1) with wave equation (2.2) is the same as the value of ξ yielding no trace anomaly, unlike what happens for Maxwell theory in Ref. [9], where the conformally invariant action is found to lead to a trace anomaly in quantum theory with mixed boundary conditions.

V. CASIMIR ENERGY AND PRESSURE

In order to evaluate the energy density ρ of our “scalar” Casimir apparatus, we project the regularized and renormalized energy-momentum tensor along a unit timelike vector $u^\mu = \left(-\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right)$. This yields

$$\begin{aligned} \rho = \langle T_{\mu\nu} \rangle u^\mu u^\nu = & -\frac{\pi^2}{1440a^4} + \frac{\pi}{7200a^4} \left[-3\pi + 6s + \frac{10}{\sin^2 s} \left(2(\pi - 2s) \right. \right. \\ & \times \left. \left. \left(-2 + \frac{3}{\sin^2 s} \right) + \cot s \left((5 + 2(\pi - s)s + 6\frac{s(-\pi + s)}{\sin^2 s}) \right) \right) \right] \varepsilon \\ & + \left(\xi - \frac{1}{6} \right) \left\{ \frac{\pi^2(2 + \cos 2s)}{8a^4 \sin^4 s} - \frac{\pi}{192a^4 \sin^5 s} \left[\left(-5 + 22(\pi - s)s \right) \cos s \right. \right. \\ & \left. \left. + \left(5 + 2(\pi - s)s \right) \cos 3s - 4(\pi - 2s)(3 \sin s + \sin 3s) \right] \varepsilon \right\}. \end{aligned} \quad (5.1)$$

The energy E stored within our Casimir cavity is given by

$$E = \int_{V_c} d^3\Sigma \sqrt{-g} \rho, \quad (5.2)$$

where $d^3\Sigma$ is the volume element of an observer with four-velocity u^μ , and V_c is the volume of the cavity. The integration in (5.2) requires the use of approximating domains, i.e. the z -integration is performed in the interval $(\zeta, a - \zeta)$, corresponding to $\frac{\pi}{a}(\zeta, a - \zeta)$ in the s variable, taking eventually the $\zeta \rightarrow 0$ limit. We thus obtain

$$E_\xi = -\frac{\pi^2 A}{1440a^3} - \frac{\pi^2 A \varepsilon}{5760a^3} + \left(\xi - \frac{1}{6} \right) \frac{\pi A}{4a^3} \left(1 + \frac{\varepsilon}{4} \right) \lim_{\zeta \rightarrow 0} \frac{\cos \zeta}{\sin^3 \zeta}, \quad (5.3)$$

where A is the area of parallel plates. Note that the conformal coupling value $\xi = \frac{1}{6}$ is picked out as the only value of ξ for which the Casimir energy remains finite. In this case, reintroducing the constants \hbar, c and writing explicitly ε , we find

$$E_c = -\frac{\hbar c \pi^2}{1440} \frac{A}{a^3} \left(1 + \frac{1}{2} \frac{ga}{c^2} \right). \quad (5.4)$$

In the same way, the pressure P_ξ on the parallel plates is found to be [9]

$$P_\xi(z=0) = \frac{\pi^2}{480a^4} + \frac{\pi^2 \varepsilon}{1440a^4} - \left(\xi - \frac{1}{6} \right) \frac{\pi \varepsilon}{16a^4} \lim_{s \rightarrow 0} \frac{\cos s}{\sin^3 s}, \quad (5.5)$$

$$P_\xi(z=a) = -\frac{\pi^2}{480a^4} + \frac{\pi^2\varepsilon}{1440a^4} + \left(\xi - \frac{1}{6}\right) \frac{\pi\varepsilon}{16a^4} \lim_{s \rightarrow \pi} \frac{\cos s}{\sin^3 s}. \quad (5.6)$$

Once again, one can get rid of divergent terms by setting $\xi = \frac{1}{6}$, which leads to

$$P_c(z=0) = \frac{\pi^2}{480} \frac{\hbar c}{a^4} \left(1 + \frac{2}{3} \frac{ga}{c^2}\right), \quad (5.7)$$

$$P_c(z=a) = -\frac{\pi^2}{480} \frac{\hbar c}{a^4} \left(1 - \frac{2}{3} \frac{ga}{c^2}\right). \quad (5.8)$$

VI. CONCLUDING REMARKS

In our paper we have obtained, for the first time in the literature, a detailed evaluation of the regularized and renormalized energy-momentum tensor for a scalar Casimir apparatus in a weak gravitational field, the plates being parallel plates upon which a real massless scalar field is required to obey homogeneous Dirichlet conditions. Moreover, the trace anomaly is found to vanish provided the free scalar field is conformally coupled to gravity. The conformal coupling also ensures a finite Casimir energy and finite pressure on the plates.

The electromagnetic case is definitely more involved: the boundary conditions are a mixture of Dirichlet and Robin conditions, and if the Casimir apparatus is set in a weak gravitational field the trace anomaly is not found to vanish [9], despite that the classical Maxwell action is conformally invariant in four spacetime dimensions. At least three investigations are now in order:

- (i) To repeat the scalar analysis with Robin boundary conditions, to understand whether the latter are responsible for the nonvanishing trace anomaly found in Ref. [9] for the electromagnetic case.
- (ii) To work out the relation (if any) between our small- ε asymptotics of the Feynman Green function G_F , with the following limit as $x' \rightarrow x$, and the usual approach where one first considers the Schwinger–DeWitt asymptotics of G_F at small values of the world function [16, 20, 21]. The two approaches are not obviously equivalent nor easily comparable, but the novel features found in Ref. [9] make it compelling to produce further efforts along both lines.
- (iii) To relate our results to the enlightening energy-momentum analysis in Ref. [22].

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